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COMPARING THE BIFURCATION OF PERIODIC SOLUTIONS FOR GENERIC DIFFERENTIAL EQUATIONS

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ABSTRACT

As we consider equivariant differential equations that can be solved by specific finite groups. A system of non-linear functional differential equations (NFDEs) cannot be solved using an operator that compactly smoothest the beginning data over time, in contrast to delayed functional differential equations. Both the point spectrum and the Poincare operator's essential spectrum stabilize the periodic solution. We show that a periodic solution's essential spectrum can undergo a 'normal' bifurcation while crossing the unit circle, altering its stability.

Keywords: Hopf bifurcation, normal form theory, center manifold theory, degree theory.

I. INTRODUCTION

For symmetrical systems, periodic solutions behave differently when bifurcate. It could be feasible to do a direct assessment of the bifurcation in the absence of symmetry for certain model equations. However, "Singularities and Groups in Bifurcation Theory II" warns that symmetry may make it difficult to highlight specific analysis. The bifurcation analysis can be accelerated by symmetry. A system's symmetry can simplify bifurcation analysis by reducing it to fixed-point subspaces, employing invariant theory, and using equivariant singularity theory. [1].

By imposing very multiplicative eigenvalues on the system, symmetry may often make bifurcation analysis more difficult. It may be useful to divide the data into isotypic subgroups when the data is symmetrical. Research into periodic solutions has led to the development of a symmetric theory of Hopf bifurcation, which suggests a model-independent approach to bifurcation prediction.

The Equivariant Hopf Theorem, a major result of this field, asserts that given any two-dimensional fixed-point subspace, there exist isotropy subgroups that are comparable to periodic solutions to equations involving a symmetry group. Periodic solutions are rational because... When researching periodic solutions, for example, it is crucial to understand how limit sets function. The presence of a negative real component in all eigenvalues of the linearization is a well-known condition for asymptotic stability, which is also called linear stability. According to the standard theorem, a periodic solution needs to be linearly stable before it can be said to be asymptotically stable. When the real part of an eigenvalue moves away from zero along the complex plane's imaginary axis, a periodic solution splits. [2].

This makes it hard to properly attribute results about bifurcation to model parameters and to compare and assess studies on the phenomenon. A variety of practical and general frameworks and methodologies were assembled by Guo and Wu in their book, which is helpful for parameter analysis of bifurcation occurrences in first-order differential equations (FDEs).

when it comes to differential equations, bifurcation theory is essential for understanding how solutions' behavior varies as parameters are changed. Particularly interesting is the study of periodic solution bifurcation, which investigates the emergence, evolution, and disappearance of periodic orbits as a function of parameter changes. The basic ideas, kinds of bifurcations, and mathematical frameworks used to study periodic solutions to general differential equations are intended to be shed light upon in this introductory discussion of bifurcation. When a parameter is altered, the structure of the solutions to a system might change qualitatively; this is called bifurcation. When discussing differential equations, this usually implies that the stability or number of equilibrium points or periodic orbits changes. When bifurcation theory is applied to periodic solutions, it finds out how tiny parameter changes may create or destroy periodic orbits, or modify their stability. This area is fundamental for comprehending a wide range of manmade and natural systems that exhibit periodic behavior, including mechanical oscillations, electrical circuits, and biological rhythms.

The Hopf bifurcation, about the emergence of a periodic solution from an equilibrium point when a parameter passes a critical threshold, is one of the most researched and one of the simplest bifurcation types. A supercritical Hopf bifurcation arises from a stable equilibrium with a stable periodic orbit, while a subcritical bifurcation gives rise to an unstable periodic orbit with complicated dynamic activity, frequently chaos, as an accompanying feature. In order to determine when these bifurcations take place, the Hopf bifurcation theorem uses the eigenvalues of the system's Jacobian matrix at equilibrium. Among the many important types of limit cycles is the saddle-node bifurcation. In this type, two periodic solutions, one stable and one unstable, clash and destroy each other when a parameter is adjusted. Homoclinic and heteroclinic bifurcations, in which periodic orbits may couple to saddle points in phase space, resulting in complex and often unexpected dynamics, are conceptually similar to this phenomena. More complicated and rich behavior may be produced by higher-dimensional systems via the interaction of many periodic orbits and equilibria, which makes these bifurcations all the more important [3].

There are more complex cases of bifurcations in periodic solutions than these classical ones, such as period-doubling bifurcations, in which an orbit with one period splits into an orbit with two periods, or twice the original period. Because a cascade of very irregular, non-periodic behavior may be produced by repeated period-doubling, this kind of bifurcation is an important pathway to chaos. A broad variety of physical and biological systems have shown this path to chaos, first proposed by Feigenbaum, which is defined by a universal scaling characteristic. Strong mathematical methods and tools are needed for the analysis of these bifurcations. A popular method involves using normal forms, which take the system close to the bifurcation point and simplify it into a canonical form while keeping the important dynamics. Because of this simplification, the bifurcation behavior can be better studied and understood. If one wants to know how complex systems' bifurcation structures work, they may use numerical continuation techniques to monitor how periodic solutions change as parameters are changed. To further analyze the local behavior of a bifurcation, center manifold theory is essential for lowering the dimensionality of the system close to the bifurcation point. Analyzing the bifurcation in a lower-dimensional context allows researchers to retain crucial information about the stability and periodic behavior of the system by zeroing in on the center manifold, which captures the dynamics of the critical modes. Knowing how bifurcations of periodic solutions work has real-world applications in many fields. For engineers, it's useful for creating steady-state oscillatory devices like mechanical resonators and electronic oscillators. Circadian cycles and heartbeats are two examples of biological rhythms that this finding elucidates. Bifurcation theory provides an ecological framework for understanding population dynamics and the factors that trigger recurrent epidemics or mass extinctions. The study of periodic solutions bifurcating for general differential equations is an extensive and diverse area that encompasses both theoretical analysis and real-world applications. One way to better understand the dynamic behavior of complex systems is to look at how periodic behavior develops and changes as parameters are changed. We will delve further into certain bifurcation types, mathematical methodologies, and their applications in different scientific and engineering disciplines after this initial introduction. The dynamic behaviors of many systems, both natural and artificial, may be better understood, controlled, and used with the help of bifurcation theory, which is still an area of active research and development [4].

II. LITERATURE REVIEW

Jie Li and Shangjiang Guo1 (2015) An introduction to the theory of bifurcations in functional differential equations is given in this paper. We begin with a brief introduction to functional differential equations and bifurcation in the context of dynamical systems. We then review the literature on functional differential equation bifurcation, including Hopf bifurcation, normal form theory, Lyapunov-Schmidt reduction, degree theory, and all other relevant issues. [5].

Gottimukkala Varma et. al. (2011) Circularly hauled cable-body systems enable cargo delivery, surveillance, and aerial and maritime vehicle towing. A very low speed for the end mass or body relative to the towing vehicle is essential for providing a solid operating platform. By using bifurcation analysis, one may also determine the lowest possible end mass radius. To demonstrate the proposed design algorithm, an example is provided [6].

Ward Melis et al (2017) We address the nonlinear BGK problem by introducing an asymptotically sound explicit high-order projective integration method. There is a clear, easy-to-follow procedure that begins with a series of smaller steps to help with the most challenging parts of the response. "The next step is to implement an arbitrary-order Runge-Kutta algorithm using the projected time derivative. We find that the number of inner time steps and the boundaries of the outer time steps are independent of the stiffness of the BGK source term by examining the spectrum of the linearized BGK operator. Here are some numerical examples in one and two dimensions to show how the method works." [7].

Bofu Wang et al (2016) "A complete numerical study of heat convection in a vertical cylindrical chamber with insulation, a heated bottom, and a cooled top is presented. While Prandtl ranges from 0.05 to 1, one remains constant as the ratio of height to radius. This research takes Rayleigh numbers up to 16,000 into account. Both normal and abnormal flow patterns can be described by one of ten distinct flow regimes. A lower Rayleigh number is required for the steady-to-oscillatory transition for low Prandtl numbers as opposed to large Prandtl numbers. Two distinct flow regimes can coexist for each given parameter configuration, according to a bifurcation analysis. For a Prandtl number of 1, we investigate the impact of flow structure on heat transfer." [8].

Maoan Han et al (2012) In Chapter 2, we discussed the computation of normal forms. A sweeping approach is first shown, which integrates normal form computation with center manifold theory. Following that, we go further into a perturbation method that has shown computational efficacy [9].

III. THEORETICAL ASPECTS

The theory of bifurcation of periodic solutions in generic differential equations, a deep mathematical field, explains how seemingly tiny changes in system parameters can impact system behavior. The idea of bifurcation is central to this theory; it signifies, in general, how the qualitative or topological structure of a system's solutions changes as a parameter is modified. Periodic orbits, which are closed paths in phase space, are studied in bifurcation theory in relation to periodic solutions. The theory aims to understand how these changes cause periodic orbits to arise, vanish, or alter stability. The Hopf bifurcation, first proposed by the German mathematician Eberhard Hopf, is a key concept in the science of bifurcations. The Hopf bifurcation theorem states that when a parameter approaches a critical value, the system of differential equations may transition from a stable equilibrium to a periodic solution. After linearizing the system around the equilibrium point, one typical way to understand this transition is to look at the eigenvalues of the Jacobian matrix. A Hopf bifurcation happens when two complex conjugate eigenvalues move from the left half-plane to the right, crossing the imaginary axis [10]. An unstable periodic orbit arises in the subcritical situation, while a tiny stable periodic orbit forms in the supercritical case, according to the theorem. Bifurcation of limit cycles at the saddle-node when two periodic orbits, one stable and one unstable, encounter and destroy each other when a parameter is adjusted is another important theoretical issue. The idea of homoclinic and heteroclinic bifurcations is strongly connected to this phenomena, which may cause sudden shifts in the dynamics of the system. The complicated and often chaotic dynamics of homoclinic bifurcations are generated when periodic orbits encounter saddle points. Likewise, heteroclinic bifurcations enhance the dynamical behavior of the system by establishing linkages between several saddle points. Another important theoretical component is the study of period-doubling bifurcations, often known as flip bifurcations. An unstable periodic orbit may split in two at a period-doubling bifurcation, creating an orbit with twice the initial period. This cycle may keep on until there is complete disorder due to a domino effect of period doublings. Mitchell Feigenbaum has explored the period-doubling approach to chaos in detail, and it is defined by a universal scaling rule. Underscoring the profound interconnections across apparently unrelated systems, this universality shows that the ratio of succeeding bifurcation intervals approaches a constant, the Feigenbaum constant [11].

Analyzing these bifurcations mathematically calls for strong methods and instruments. A key contribution of normal form theory to the simplification of bifurcation studies is the reduction of the system surrounding the bifurcation point to a canonical form. The essential dynamics are preserved in this simplified version, which also eliminates unnecessary details. After the initial system is normalized, researchers can verify the stability of the emerging periodic solutions and dig deeper into the bifurcation behavior. Center manifold theory is another useful tool for dimensionality reduction at the bifurcation point of the system. The system's dynamics can be captured by using a lower-dimensional manifold, the center manifold, which contains the critical modes in charge of the bifurcation, as stated in the center manifold theorem. This reduction allows for a more manageable analysis of the local behavior around the bifurcation, which improves our understanding of the system's stability and periodic dynamics. The study of bifurcations of periodic solutions relies heavily on numerical continuation methods, which are complementary to these analytical approaches [12]. In order to find bifurcation points and map the bifurcation structure of the system, these approaches use numerical tracking of solutions as parameters are varied. When solving complicated problems analytically proves to be impossible, numerical continuation might shed light on the problem. In addition to these more traditional methods, contemporary advancements in bifurcation theory draw on concepts from dynamical systems theory and topology. An important tool for understanding the complex dynamics at work in the emergence and evolution of periodic solutions is the notion of bifurcation diagrams, which visually depict the changes in the system's solutions as parameters change. The intricate interaction between several dynamical behaviors is commonly shown by bifurcation diagrams, which can show a rich tapestry of stable and unstable solutions.

Results from bifurcation theory have far-reaching consequences in many branches of science and engineering. Circadian rhythms, heartbeats, and brain oscillations are all examples of rhythmic processes in biological systems that may be better understood by gaining an appreciation for bifurcations. Stable oscillatory systems, where the resilience and stability of periodic behavior are vital, are designed with the help of bifurcation analysis in engineering. Particularly relevant here are mechanical resonators and electronic circuitry. By offering a framework for the prediction and management of ecological systems, bifurcation theory helps scientists comprehend population dynamics and the factors that cause periodic outbreaks or extinctions. More specifically, the aim of control theory is to obtain desired dynamical behaviors by manipulating system characteristics, which is where bifurcation theory meets with control theory. The establishment of desired behaviors, like steady oscillations in biological or chemical reactors, or the prevention of unwanted ones, like the start of chaos in engineering systems, may be achieved by controlling bifurcations [13]. Bridging the gap between theoretical bifurcation analysis's abstract mathematical insights and real-world applications, this intersection highlights the practical value of the field. A complicated and extensively researched theoretical subject is the bifurcation of periodic solutions to general differential equations. Providing a thorough framework for understanding the emergence and evolution of periodic behavior in complex systems, bifurcation theory encompasses both basic and sophisticated methodologies, ranging from Hopf bifurcations and period-doubling cascades to normal forms and center manifold theory. Mathematical rigor and numerical approaches allow scientists to decipher the complex dynamics of periodic solutions, which in turn reveal the processes that cause technological and natural rhythmic events. Bifurcation theory continues to have a significant and positive influence on many fields of science and engineering, thanks to its theoretical underpinnings that enhance our knowledge of dynamical systems and guide practical applications in many more [14].

The initial value problem

A solution $x(t)$ of (1) on $t \in [0, \infty)$ has to be named before a function's initial conditional element may be defined,

$$x(\theta) = x_0(\theta), \ -\tau \leq \theta \leq 0,$$

$$x_0 \in C([-\tau, 0], \mathbb{R}^n),$$

"Banach space of continuous bounded functions with a bounded derivative is the space to which C maps. $[-\tau, 0]$ into $\mathbb{R}^n$. . Even if both f and x_0 x0 are infinitely smooth, although there may be breaks in the first derivative of the solution x(t). This is because"

$$\left.\frac{dx_0(\theta)}{d\theta}\right|_{\theta=0^-}$$

"The right side of (1) is typically not zero when t = 0. In contrast to RFDEs, where the solution profile smoothes over time, the dependence of f on maintains the derivative's discontinuities. That adds another layer of complexity to the analysis since the first derivative of x(t) can be discontinuous for every t = kr with k N. Discontinuities in the derivatives of x0 will propagate in the same way as discontinuities in x. On the other hand, periodic solutions can have their continuity features checked under certain circumstances." [15].

IV. STABILITY ANALYSIS

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The solution z(t) to is a periodic function with period.

$$z(t+T) = z(t), \ \forall t.$$

8t: The shape of the variational equation that characterizes this solution is a neutral functional differential equation in n dimensions [16].

$$\dot{y}(t) = A(t)y(t) + B(t)y(t-\tau) + C(t)\dot{y}(t-\tau).$$

Here, using $f \equiv f(u, v, w, \lambda)$,

$$A(t) \triangleq \frac{\partial f}{\partial u}(z(t), z(t-\tau), \dot{z}(t-\tau), \lambda),$$
$$B(t) \triangleq \frac{\partial f}{\partial v}(z(t), z(t-\tau), \dot{z}(t-\tau), \lambda)$$

and

$$C(t) \triangleq \frac{\partial f}{\partial w}(z(t), z(t-\tau)\dot{z}(t-\tau), \lambda),$$

$$A(t+T) = A(t), \ B(t+T) = B(t), \ C(t+T) = C(t), \ \forall t.$$

Spectrum of the solution operator

One possible rewriting of the variational equation is as, where z(t) is the continuously differentiable periodic solution. [17].

$$(d/dt)[y(t) - C(t)y(t-\tau)] = A(t)y(t) + (B(t) - \dot{C}(t))y(t-\tau).$$

is a particular case of the eq

$$(d/dt)[y(t) - G(t, y_t)] = L(t, y_t),$$

where

$G : \mathbb{R} \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ and $L : \mathbb{R} \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$) ! “There are two continuous arguments (t and t + 1) to Rn; the second argument is linear in Rn and the third is T-periodic. The existing Floquet theory is restricted to NFDEs since a complete generalization to such systems is not feasible. Poincare's theorem, linearized $S_L(t+T, t)$ t+T ; t) involves adding a compact operator to a contraction operator. as a result, the spectrum $S_L \ (\sigma(S_L))$ comprises a point spectrum p(SL) and an essential spectrum e(SL) that ranges from zero to one. It follows that e(SL) must exist since G(t; yt) is a "neutral" term in (4). Similar to the solution operator SD(t + T; t) for the difference equation, the fundamental spectrum of SL is also same.” [18].

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$$y(t) - G(t, y_t) = 0,$$

or, equivalently, e(SL) = (SD) [6]. "The SL point spectrum tends to zero or near the essential spectrum asymptotically. Bear in mind that the solution operator SL is tiny and has a point spectrum where zero is the only cluster point for RFDEs." [19].

V. FLOQUET THEORY

The stability of the difference operator D(t; yt) in the theory of these types of NFDEs,

$$D(t, y_t) \equiv y(t) - G(t, y_t),$$

performs a key role. If the difference equation is linear in yt, continuous in both arguments, and has a zero solution, then the operator is stable.

$$D(t, y_t) = 0, \ t \geq 0,$$

"As the complex integer Floquet multiplier of z(t), t is an ordinary eigenvalue of the operator SL(t + T; t). If an eigenvalue of SL(t + T; t) has a generalized eigenspace with finite dimensions, we say that it is normal. The fact that the Floquet multipliers are independent on t is demonstrated. For stable D outside of an origin-centered disk with radius r > re, the number of Floquet multipliers in the SL spectrum is finite. These Floquet multipliers are commonly used to explain the (local) stability properties of the periodic solution z(t)" [20].

Let $m(\epsilon) \in \mathbb{N}$ represent the Floquet multipliers of $S_L(T, 0)$ with modulus greater than $r_e + \epsilon, \ \epsilon > 0.$ Since as tends to zero, is a nondecreasing function, we may either

$$\lim_{\epsilon \to 0^+} m(\epsilon) = \infty,$$

In this scenario, we refer to the essential spectrum as being "invisible from the outside" since p(SL) asymptotes to e(SL) in the limit of large SL. Also, we may

$$\lim_{\epsilon \to 0^+} m(\epsilon) = m^* \in \mathbb{N},$$

"A 'normal' bifurcation in the branch stability is achieved when the Floquet multipliers along the branch either cross the unit circle in the complex plane or the radius of the essential spectrum recrosses 1/2. By hiding the crucial spectrum, we expect D to go through an endless loop of 'normal' bifurcations until it breaks down. The number of unstable modes increases from any finite value by an infinitesimal amount if the essential spectrum is visible from the outside. If the delays depend non-continuously on one another, then the spectral radius of SD for steady-state solutions to the multiple-delay situation is z(t) x0. Our numerical results suggest a similar dependence with respect to the delay and the time T; nonetheless, we will revisit this topic in the future."

VI. CONCLUSION

A numerical bifurcation analysis was conducted and the results were provided for the given system of NFDEs. Preexisting NFDE theory is intricate and open to ongoing refinement. The way neutral and retarded FDEs behave in solutions is very different from one another. Over time, the original data is no longer averaged and the solution operator for retarded FDEs ceases to be a compact operator. Where the Poincar e operator's essential spectrum (e(SL)) is and how many dominating Floquet multipliers there are determine periodic solution stability. Nobody has studied periodic NFDE solutions' numerical behavior when they bifurcate. We demonstrate that NFDEs are unique by computing periodic solutions, ensuring their stability, and showing that periodic solution branches can continue using the Newton-Picard method. We show that e(SL) crossing the unit circle can change the stability of a periodic solution and cause a 'normal' bifurcation while continuation is underway, highlighting its importance in stability analysis. Thus, the e(SL) spectral radius re must be checked continuously. Re may be discontinuous along a branch at "resonance" locations, and time and delay are interdependent, adding complication. We have maximum and lower limits on re outside of these "resonance" regions. We compare computational and analytical conclusions whenever possible and raise several important yet unsolved questions during our inquiry.

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