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ANALYZING SOLUTIONS OF VOLTERRA AND FREDHOLM INTEGRAL EQUATIONS

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ABSTRACT

The analysis of solutions for Volterra and Fredholm integral equations is a pivotal area of study within the broader field of integral equations, which finds applications across various disciplines such as physics, engineering, and mathematical biology. Volterra integral equations, characterized by their lower integration limit being variable, often model systems with memory effects, such as population dynamics and viscoelastic materials. Conversely, Fredholm integral equations have fixed integration limits and are instrumental in solving problems related to potential theory, quantum mechanics, and boundary value problems. The methods for solving these integral equations include analytical approaches like the method of successive approximations and the Neumann series, as well as numerical techniques such as quadrature methods, iterative methods, and spline approximations. The primary challenge in analyzing these equations lies in ensuring the existence and uniqueness of solutions, as well as in developing stable and efficient computational algorithms. By exploring both theoretical and practical aspects, researchers aim to derive conditions under which solutions exist and converge, thereby advancing our understanding and capability to address complex real-world problems modeled by these integral equations.

Keywords: - Algorithms, Problem, Integral, Equation, Method.

I. INTRODUCTION

Analyzing the solutions of Volterra and Fredholm integral equations constitutes a fundamental pursuit in mathematical analysis, with far-reaching implications across diverse scientific domains. Integral equations serve as powerful tools for modeling a wide array of phenomena characterized by continuous interactions, memory effects, and boundary conditions. Volterra integral equations, named after the Italian mathematician Vito Volterra, encompass systems where the lower limit of integration is variable. They find applications in various fields such as population dynamics, biology, economics, and physics, particularly in problems involving systems with memory or delay effects. Fredholm integral equations, on the other hand, named after the Swedish mathematician Erik Ivar Fredholm, have fixed integration limits and are often utilized to describe problems related to potential theory, quantum mechanics, and boundary value problems. The study of these integral equations involves understanding their properties, existence, uniqueness, and behavior of solutions, as well as developing efficient numerical algorithms for their solution. This comprehensive investigation spans both theoretical and computational realms, delving into the intricacies of functional analysis, operator theory, and numerical methods. By unraveling the complexities inherent in these equations, researchers endeavor to unveil the underlying principles governing real-world phenomena and to devise effective strategies for tackling a myriad of scientific and engineering challenges. Thus, the analysis of Volterra and Fredholm integral equations stands at the nexus of theoretical exploration and practical application, driving innovation and progress in various scientific disciplines.

II. REVIEW OF LITERATURE

Aziz, Imran et Al., (2013) wavelet Fredholm integral equations Volterra integral equations First-order integro-differential equations Second-order integro-differential equations Fourth-order integro-differential equations abstract In this paper, a novel technique is being formulated for the numerical solution of integral equations (IEs) as well as integro-differential equations (IDEs) of first and higher orders. The present approach is an improved form of the Haar wavelet methods (Aziz and Siraj-ul-Islam, 2013, Siraj-ul-Islam et al., 2013). The proposed modifications resulted in computational efficiency and simple applicability of the earlier methods (Aziz and Siraj-ul-Islam, 2013, Siraj-ul-Islam et al., 2013). In addition to this, the new approach is being extended from IDEs of first order to IDEs of higher orders with initial-and boundary-conditions. Unlike the methods (Aziz and Siraj-ul-Islam, 2013, Siraj-ul-Islam et al., 2013) (where the kernel function is being approximated by two-dimensional Haar wavelet), the kernel function in the present case is being approximated by one-dimensional Haar wavelet. The modified approach is easily extendable to higher order IDEs. Numerical examples are being included to show the accuracy and efficiency of the new method.

Borhan, J. et al., (2023) This research work demonstrates an approach to solve nonlinear evolution equations via the generalized \(\left(G{\prime }/G\right)\)-expansion method which is an advantageous mathematical tool for establishing abundant solutions of these types of nonlinear evolution equations. Here, we select the \(\left(1+1\right)\)-dimensional integro-differential Ito equation and \(\left(2+1\right)\)-dimensional integro-differential Sawda-Kotera equation to extract the closed traveling wave solutions by using the mentioned method. In applied Mathematics, mathematical Physics, engineering science as well as real time application fields have enormous application of these type of equations. The new traveling wave solutions derived by this method are involving hyperbolic function, trigonometric function and rational function. This method is direct, efficient, convenient and powerful to solve other nonlinear evolution equations. Moreover, the features of the solutions are illustrated by some figures.

Uwaheren, Ohigweren Airenoni et al., (2021) This paper deals with the solution of Fractional Integro-differential Equations of Fredholm type using Legendre Galerkin Method. The concept of Legendre Galerkin Method was implemented on some examples of fractional integro-differential equations of Fredholm type to illustrate the practicability of the method. Fractional derivatives of Caputo sense were used throughout the paper. The results obtained show that the method is reliable and accurate for the kind of problems considered when compared to the exact solutions.

Ishak, Fuziyah et al., (2019) Fuzzy differential equations (FDEs) play important roles in modeling dynamic systems in science, economics and engineering. The modeling roles are important because most problems in nature are indistinct and uncertain. Numerical methods are needed to solve FDEs since it is difficult to obtain exact solutions. Many approaches have been studied and explored by previous researchers to solve FDEs numerically. Most FDEs are solved by adapting numerical solutions of ordinary differential equations. In this study, we propose the extended Trapezoidal method to solve first order initial value problems of FDEs. The computed results are compared to that of Euler and Trapezoidal methods in terms of errors in order to test the accuracy and validity of the proposed method. The results shown that the extended Trapezoidal method is more accurate in terms of absolute error. Since the extended Trapezoidal method has shown to be an efficient method to solve FDEs, this brings an idea for future researchers to explore and improve the existing numerical methods for solving more general FDEs.

Singh, Somveer et al., (2017) In this paper, we propose and analyze an efficient matrix method based on shifted Legendre polynomials for the solution of non-linear volterra singular partial integro-differential equations(PIDEs). The operational matrices of integration, differentiation and product are used to reduce the solution of volterra singular PIDEs to the system of non-linear algebraic equations. Some useful results concerning the convergence and error estimates associated to the suggested scheme are presented. Illustrative examples are provided to show the effectiveness and accuracy of proposed numerical method.

Brunner, Hermann. (2018). The aim of this paper is to describe the current state of the numerical analysis and the computational solution of non-standard integro-differential equations of Volterra and Fredholm types that arise in various applications. In order to do so, we first give a brief review of recent results concerning the numerical analysis of standard (ordinary and partial) Volterra and Fredholm integro-differential equations, with the focus being on collocation and (continuous and discontinuous) Galerkin methods. In the second part of the paper we look at the extension of these results to various classes of non-standard integro-differential equations type that arise as mathematical models in applications. We shall see that in addition to numerous open problems in the numerical analysis of such equations, many challenges in the computational solution of non-standard Volterra and Fredholm integro-differential equations are waiting to be addressed.

III. VOLTERRA INTEGRAL EQUATION

Within the scope of this section, we will be discussing non-homogeneous Volterra integral equations of the second derivative of the form.

$$u(x) = f(x) + \lambda \int_a^x K(x,t)F(u(t))dt,$$

In this equation, the variables a and x represent the limits of integration, λ corresponds to a constant parameter, and K(x, t) is referred to as the kernel of the integral equation. This kernel is a function of two variables, x and t. Under the integral sign, the function u(x) that will be calculated appears, and it also appears within the integral sign and outside the integral sign. In addition, it appears outside the integral sign. It is assumed that the functions f(x) and K(x, t) are already determined.

The starting value issue is the source of the Volterra integral equations. It is well-known that many scientific domains produce linear and non-linear Volterra integral equations. Some examples include semi-conductor devices, epidemic spread, and population dynamics. Volterra began experimenting with integral equations in 1884 and dove headfirst into his studies in 1896. In 1888, du Bois-Reymond bestowed the term "integral equation" upon the concept. The term Volterra integral equation, however, was initially used by Lalesco in 1908. Abel thought about the challenge of finding the equation of a vertical plane curve. The time it takes for a mass point to fall to the horizontal from a specific positive height along this curve is a function of the height that is defined in this issue.

IV. FREDHOLM INTEGRAL EQUATION

A wide variety of scientific contexts give rise to Fredholm integral equations, including those arising from boundary value issues. The contributions of Erik Ivar Fredholm (1866–1927) to spectral theory and integral equations are what have made him most famous. The Swedish mathematician Fredholm laid the groundwork for operator theory with his seminal work in integral equation theory and his publication in Acta Mathematica. In an integral equation, the unknown function u(x) takes on an integral form. A common form for general integral equations in u(x) is

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \qquad (3.2.1)$$

The constant parameters λ and the function K(x, t) which takes two variables x and t and is known as the kernel of the integral equation are included, together with the constant limits of integration a and b. The determined function u(x) is visible both within and outside the integral sign; it also appears under the sign. The functions K(x, t) and f(x) are predefined.

Description of the Methods

The Adomian Decomposition Method (ADM), the Modified Adomian Decomposition Method (MADM), the Variational iteration (VIM) Method, and the Homotopy Perturbation Method

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(HPM) are just a few examples of the innovative approaches that have been working to improve upon previously established and effective strategies for Fredholm essential conditions.

Adomian Decomposition Method(ADM)

Using an infinite series, the unknown function u(x) may be defined using the Adomian decomposition approach.

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{3.2.2}$$

This often involves the repeated determination of the components un(x). A decomposition of the non-linear operator F(u) into an infinite series of polynomials may be expressed as

$$F(u) = \sum_{n=0}^{\infty} A_n, \tag{3.2.3}$$

where An are defined as the so-called Adomian polynomials of u0, u1,..., un,

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0} \quad , n = 0, 1, 2, \ldots \tag{3.2.4}$$

or equivalently

$$\begin{aligned}
A_0 &= F(u_0), \\
A_1 &= u_1 F'(u_0), \\
A_2 &= u_2 F'(u_0) + \frac{1}{2}u_1^2 F(u_0), \\
A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_o) + \frac{1}{3}u_1^3 F'''(u_0),
\end{aligned} \tag{3.2.5}$$

The fact that certain techniques may be used to construct these polynomials for any class of nonlinearity is now widely recognized. Just recently, Wazwaz came out with a different way to build Adomian polynomials.

$$\sum_{i=0}^{\infty} u_i(x) \;=\; f(x) + \lambda \sum_{i=0}^{\infty}\left(\int_a^b K(x,t)A_i(t)dt\right)$$

The components u0, u1, u2, · · · are usually determined recursively by

$$u_0 = f(x),$$
$$u_1 = \lambda \int_a^b K(x,t)A_0(t)dt,$$
$$u_n = \lambda \int_a^b K(x,t)A_{n-1}(t)dt, \; n \geq 1.$$

(3.2.6)

Then, $u(x) = \sum_{i=0}^{n} u_i(x)$ as the approximate solution.

Modified Adomian Decomposition Method (MADM)

Equation (3.2.9) using the Fredholm integral is often solved using the Adomian decomposition method.

$$G(u(x)) = \sum_{n=0}^{\infty} A_n,$$

(3.2.7)

where An, n ≥ 0 are the Adomian polynomials determined formally as follows

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\mu^n} G(\sum_{i=0}^{\infty} \mu^i u_i)\right]\Big|_{\mu=0}.$$

(3.2.8)

The solution of u is represented by the following series using the conventional decomposition approach.

$$u(x) = \sum_{i=0}^{\infty} u_i(x).$$

(3.2.9)

By substituting (3.2.7) and (3.2.9) in eq (3.2.1) we have

$$\sum_{i=0}^{\infty} u_i(x) = f(x) + \lambda \sum_{i=0}^{\infty} \left(\int_a^b K(x,t)A_i(t)dt \right)$$

Wazwaz spoke about the MADM. Partially dividing the function f(x) into f1(x) and f2(x) is the underlying assumption of this method. Assuming this to be true, we decided to

$$f(x) = f_1(x) + f_2(x).$$

(3.2.10)

When the function f is multi-participant and may be split into two components, we use this decomposition. Here, f is often a polynomial plus a transcendental or trigonometric function added together. Making the right decision for the function f1(x) is critical.

We choose f1(x) as one term of f or, if feasible, a number of terms, and f2(x) as the remainder of f so that the procedure may be applied gradually. With the help of the MADM, we may rewrite equation (3.2.1) as shown in (3.2.10).

$$u(x) = f_1(x) + f_2(x) + \lambda \int_a^b K(x,t)G(u(t))dt.$$

The components u0, u1, u2, · · · are usually determined recursively by

$$\begin{aligned} u_0 &= f_1(x), \\ u_1 &= f_2(x) + \lambda \int_a^b K(x,t)A_0(t)dt, \\ u_n &= \lambda \int_a^b K(x,t)A_{n-1}(t)dt, \ n \geq 1. \end{aligned} \tag{3.2.11}$$

Then, $u(x) = \sum_{i=0}^{n} u_i(x)$ as the approximate solution.

V. CONCLUSION

The analysis of Volterra and Fredholm integral equations is a multifaceted endeavor essential for understanding and solving a wide range of problems across numerous scientific disciplines. Through theoretical exploration and computational techniques, researchers unravel the intricacies of these equations, shedding light on fundamental principles governing real-world phenomena. By bridging theory and application, this research facilitates advancements in fields as diverse as physics, biology, engineering, and economics, offering solutions to complex challenges and driving innovation forward. Ultimately, the study of Volterra and Fredholm integral equations stands as a testament to the power of mathematical analysis in shaping our understanding of the world and advancing scientific knowledge.

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